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**Results in Mathematics**

# Generalized Pexider Equation on an Open Domain

Małgorzata Chudziak and Barbara Sobek

**Abstract.** Inspired by the papers by Abbas, Aczél and by Chudziak and Tabor, we consider the problem of existence and uniqueness of extensions for the generalized Pexider equation

$$k(x+y)=l(x)+m(x)n(y) \quad \text{for} \quad (x,y)\in D,$$

where $D$ is a nonempty open subset of a normed space. We show that the connectedness of $D$, assumed in the mentioned above papers, can be weakened.

**Mathematics Subject Classification.** 39B52, 39B82.

**Keywords.** Generalized Pexider equation, Restricted domain, Extension, Additive function, Exponential function.

## 1. Introduction

One of the most important questions concerning the solutions of the classical Pexider equation

$$f(x+y)=g(x)+h(y), \tag{1}$$

is the problem of existence and uniqueness of extension. The well known result of Radó and Baker [11] states that if $X$ is a linear topological space, $D$ is a nonempty open and connected subset of $X^2$, $Y$ is an Abelian group and a triple of functions $(f,g,h)$, where $f:D_+\to Y$, $g:D_1\to Y$, $h:D_2\to Y$, with

$$D_1:=\{x\in X \mid (x,y)\in D \text{ for some } y\in X\},$$
$$D_2:=\{y\in X \mid (x,y)\in D \text{ for some } x\in X\}$$

and

$$D_+:=\{x+y \mid (x,y)\in D\},$$

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satisfies Eq. (1) for all pairs $(x, y) \in D$, then there exists a unique extension of $(f, g, h)$ to the solution $(F, G, H)$ of

$$F(x+y) = G(x) + H(y) \quad \text{for} \quad (x, y) \in X^2. \tag{2}$$

More precisely, there exists a unique triple of functions $(F, G, H)$, mapping $X$ into $Y$, satisfying (2) and such that $f = F_{|D_+}$, $g = G_{|D_1}$ and $h = H_{|D_2}$. The result of Radó and Baker has been substantially generalized by Forti and Paganoni [7]. They have proved (cf. [7, Theorem 5]) that in order to get the above assertion it suffices to assume that $D$ is nonempty, open and at least two of the sets $D_+$, $D_1$ and $D_2$, are connected.

The problem of extension for the following generalization of (1)

$$k(x+y) = l(x) + m(x)n(y) \quad \text{for} \quad (x, y) \in D, \tag{3}$$

in the class of quadruples $(k, l, m, n)$, where $k : D_+ \to \mathbb{R}$, $l, m : D_1 \to \mathbb{R}$ and $n : D_2 \to \mathbb{R}$, has been investigated by Aczél [4,5]. Equation (3), as well as some of its particular cases, play a crucial role in solving various problems in utility theory and decision analysis. In order to recall briefly one of them, suppose that a decision maker, having a continuous strictly increasing utility function $u$, values a lottery $L$, being a finitely-valued random variable on a given probability space, by its certainty equivalent defined by $C(L) = u^{-1}(Eu(L))$. Let $T \subset \mathbb{R}$ be a nonempty set of admissible shifts. If the equality $C(L+t) = C(L) + t$ holds for every lottery $L$ and every $t \in T$, then the decision maker is said to satisfy the delta property. The notion of delta property has been introduced by Howard [9] and Raiffa [12]. It turns out (cf. [2, Proposition 3]) that the delta property holds if and only if there exist functions $m, l : T \to \mathbb{R}$ such that the triple $(u, m, l)$ satisfies equation

$$u(x+t) = l(t) + m(t)u(x) \quad \text{for} \quad (x, t) \in \mathbb{R} \times T. \tag{4}$$

For more details concerning further applications we refer to [1,3] and to a survey paper [2]. Furthermore, the particular case $n = k$ of (3) was used to prove that the power means and the geometric mean are the only homogeneous quasiarithmetic means (cf. [8]).

In [5] it has been proved that if $D$ is an open and connected subset of a real plane, $k$ is locally nonconstant (i.e. it is nonconstant on any interval of positive length) and a quadruple of functions $(k, l, m, n)$, where $k : D_+ \to \mathbb{R}$, $l, m : D_1 \to \mathbb{R}$ and $n : D_2 \to \mathbb{R}$, satisfies Eq. (3), then there exists a unique quadruple $(K, L, M, N)$ of functions mapping $\mathbb{R}$ into $\mathbb{R}$ such that

$$K(x+y) = L(x) + M(x)N(y) \quad \text{for} \quad (x, y) \in \mathbb{R}^2 \tag{5}$$

and

$$k = K_{|D_+}, \quad l = L_{|D_1}, \quad m = M_{|D_1}, \quad n = N_{|D_2}. \tag{6}$$

Chudziak and Tabor [6] generalized this result and determined the general solution of (3) without any additional assumptions on the unknown functions $k$, $l$, $m$ and $n$. Their main results refer to the case where $X$ is a normed space and $D \subset X^2$ is nonempty, open and connected. In particular (cf. [6, Theorem

3]) they proved that, in this general setting, if $k$ is nonconstant, then every solution $(k, l, m, n)$ of (3) has a unique extension onto the solution $(K, L, M, N)$ of

$$K(x+y) = L(x) + M(x)N(y) \quad \text{for} \quad (x, y) \in X^2. \tag{7}$$

In the light of the result of Forti and Paganoni, it is a natural question, whether or not the result of Chudziak and Tabor remains true if the connectedness of $D$ is replaced by the connectedness of at least two of the sets $D_+$, $D_1$ and $D_2$. The aim of this paper is to give an answer to the above question. Note that this problem is also strictly related to the delta property. Namely, if the set $T$ of all admissible shifts is open but disconnected, then (4) is a Pexider type equation on an open disconnected subset of $\mathbb{R}^2$.

Throughout the paper, $X$ is a normed space. Recall that a function $f : X \to \mathbb{C}$ is called *additive* if $f(x+y) = f(x) + f(y)$ for $x, y \in X$, and it is called *exponential* if $f(x+y) = f(x)f(y)$ for $x, y \in X$. It is well known that if $f : X \to \mathbb{C}$ is a nonzero exponential function then $f(X) \subset \mathbb{C} \setminus \{0\}$ and $f(0) = 1$. Furthermore, if $f : X \to \mathbb{C}$ is an additive or exponential function which is constant on a nonempty and open subset of $X$ then $f$ is constant. In other words: every nonzero additive and every nonconstant exponential mapping of $X$ into $\mathbb{C}$ is locally nonconstant.

## 2. Auxiliary results

**Lemma 2.1.** *Assume that $U$ is a nonempty open subset of $X$, $A_1, A_2 : X \to \mathbb{C}$ are additive functions, $\alpha_1, \alpha_2 \in \mathbb{C}$ and*

$$A_1(x) + \alpha_1 = A_2(x) + \alpha_2 \quad \textit{for} \quad x \in U. \tag{8}$$

*Then $A_1 = A_2$ and $\alpha_1 = \alpha_2$.*

*Proof.* Since $A_1$ and $A_2$ are additive, so is $A_1 - A_2$. Furthermore, according to (8), $A_1 - A_2$ is constant on a nonempty open subset of $X$. Thus, $A_1 - A_2$ is constant, that is $A_1 - A_2 = 0$. Hence $A_1 = A_2$ which, together with (8), gives $\alpha_1 = \alpha_2$. □

**Lemma 2.2.** *Assume that $U$ is a nonempty open subset of $X$, $\phi_1, \phi_2 : X \to \mathbb{C}$ are nonconstant exponential functions, $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2 \in \mathbb{C}$ and*

$$\alpha_1\phi_1(x) + \beta_1 = \alpha_2\phi_2(x) + \beta_2 \quad \textit{for} \quad x \in U. \tag{9}$$

*Then $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Furthermore, if $\alpha_1 \neq 0$, then $\phi_1 = \phi_2$.*

*Proof.* Assume that $\alpha_1 \neq 0$. Then $\alpha_2 \neq 0$, because otherwise $\phi_1$ would be constant on $U$, which is impossible, as $\phi_1$ is a nonconstant exponential function. Fix an $x_0 \in U$ and let $B$ be an open ball in $X$, centered at 0, such that $x_0 + 2B \subset U$. Since $\phi_1$ is a nonconstant exponential function, there is a $b_0 \in B$ with $\phi_1(b_0) \notin \{0, 1\}$. So, making use of (9), we get

$$\alpha_1\phi_1(x + b_0) + \beta_1 = \alpha_2\phi_2(x + b_0) + \beta_2 \quad \text{for} \quad x \in x_0 + B.$$

As $\phi_1$ and $\phi_2$ are exponential, subtracting side by side equality (9) from the last equality, we obtain

$$\frac{\phi_1(x)}{\phi_2(x)} = \frac{\alpha_2(\phi_2(b_0)-1)}{\alpha_1(\phi_1(b_0)-1)} \text{ for } x \in x_0 + B. \tag{10}$$

Therefore $\frac{\phi_1}{\phi_2}$ is an exponential function constant on $x_0+B$, and so it is constant. Consequently $\frac{\phi_1}{\phi_2} = 1$, that is $\phi_1 = \phi_2$. Thus, in view (10), we get $\alpha_1 = \alpha_2$ which, together with (9), gives $\beta_1 = \beta_2$.

In order to complete the proof it is enough to note that, as $\phi_2$ is a nonconstant exponential function, if $\alpha_1 = 0$ then, by (9), $\alpha_2 = 0$, and so $\beta_1 = \beta_2$. □

**Lemma 2.3.** *Assume that $U$ is a nonempty open subset of $X$, $A : X \to \mathbb{C}$ is an additive function, $\phi : X \to \mathbb{C}$ is a nonzero exponential function, $\alpha \in \mathbb{C} \setminus \{0\}$, $\beta, \gamma \in \mathbb{C}$ and*

$$\alpha\phi(x) + \beta = A(x) + \gamma \ \text{ for } \ x \in U. \tag{11}$$

*Then $A = 0$ and $\phi = 1$.*

*Proof.* Let $a \in U$ and let $B \subset X$ be an open ball centered at zero such that $a + B \subset U$. Then, in view of (11), we get

$$\alpha\phi(a+x) + \beta = A(a+x) + \gamma \ \text{ for } \ x \in B.$$

Hence, as $A$ is additive and $\phi$ is exponential, letting $\tilde{\alpha} := \alpha\phi(a)$ and $\tilde{\gamma} := A(a) + \gamma$, we obtain

$$\tilde{\alpha}\phi(x) + \beta = A(x) + \tilde{\gamma} \ \text{ for } \ x \in B. \tag{12}$$

Note that we have also $\tilde{\alpha} \neq 0$. Setting in (12) $x = 0$, we get $\tilde{\alpha} + \beta = \tilde{\gamma}$. Thus, (12) becomes

$$A(x) = \tilde{\alpha}\left(\phi(x) - 1\right) \ \text{ for } \ x \in B. \tag{13}$$

Since $A$ is additive and $\phi$ is exponential, for every $x \in B$, we have $A(x) = 2A\left(\frac{x}{2}\right)$ and $\phi(x) = \phi\left(\frac{x}{2}\right)^2$. Therefore, as $\tilde{\alpha} \neq 0$, taking into account (13), for every $x \in B$, we obtain

$$\tilde{\alpha}\left(\phi\left(\frac{x}{2}\right)^2 - 1\right) = \tilde{\alpha}(\phi(x)-1) = A(x) = 2A\left(\frac{x}{2}\right) = 2\tilde{\alpha}\left(\phi\left(\frac{x}{2}\right) - 1\right).$$

Thus

$$\tilde{\alpha}\left(\phi\left(\frac{x}{2}\right) - 1\right)^2 = 0 \ \text{ for } \ x \in B$$

and so, as $\tilde{\alpha} \neq 0$, we get $\phi\left(\frac{x}{2}\right) = 1$ for $x \in B$. Hence, $\phi$ is an exponential function constant on $\frac{1}{2}B$. Thus, $\phi = 1$ which, together with (13), gives $A = 0$. □

**Lemma 2.4.** *Assume that $U \subset X$ is a nonempty connected set and $\{U_t : t \in T\}$ is a family of sets open in $U$ (with respect to the induced topology) such that $U = \bigcup_{t\in T} U_t$. Then, for every $x, y \in U$, there exist $n \in \mathbb{N}$ and $t_0, t_1, ..., t_n \in T$ such that $x \in U_{t_0}$, $y \in U_{t_n}$ and $U_{t_{i-1}} \cap U_{t_i} \neq \emptyset$ for $i \in \{1, ..., n\}$.*

*Proof.* Fix an $x \in U$. Let $Z$ be a set consisting of all elements $z \in U$ such that there exist $n \in \mathbb{N}$ and $t_0, t_1, ..., t_n \in T$ with $x \in U_{t_0}$, $z \in U_{t_n}$ and $U_{t_{i-1}} \cap U_{t_i} \neq \emptyset$ for $i \in \{1, ..., n\}$. It is not difficult to check that $Z$ is nonempty and closed-open in $U$. Thus from the connectedness of $U$ it follows that $Z = U$. □

**Lemma 2.5.** *Assume that $U$ is a nonempty open and connected subset of $X$ and $f : U \to \mathbb{C}$. Let $\{U_t : t \in T\}$ be a family of nonempty open sets such that $U = \bigcup_{t\in T} U_t$. Assume that, for every $t \in T$, there exist an $\alpha_t \in \mathbb{C}$ and an additive function $A_t : X \to \mathbb{C}$ such that*

$$f(x) = A_t(x) + \alpha_t \quad \text{for} \quad x \in U_t. \tag{14}$$

*Then, for every $s, t \in T$, we have $A_s = A_t$ and $\alpha_s = \alpha_t$.*

*Proof.* Let $s, t \in T$, $x \in U_s$ and $y \in U_t$. Then, according to Lemma 2.4, there exist $n \in \mathbb{N}$ and $t_0, t_1, ..., t_n \in T$ such that $x \in U_{t_0}$, $y \in U_{t_n}$ and $U_{t_{i-1}} \cap U_{t_i} \neq \emptyset$ for $i \in \{1, ..., n\}$. Hence

$$A_s(z) + \alpha_s = A_{t_0}(z) + \alpha_{t_0} \quad \text{for} \quad z \in U_s \cap U_{t_0}.$$

Since $U_s \cap U_{t_0}$ is a nonempty open set, applying Lemma 2.1, we obtain that $A_s = A_{t_0}$ and $\alpha_s = \alpha_{t_0}$. Then, we have

$$A_{t_0}(z) + \alpha_{t_0} = A_{t_1}(z) + \alpha_{t_1} \quad \text{for} \quad z \in U_{t_0} \cap U_{t_1}$$

and so, as previously, we conclude that $A_{t_1} = A_{t_0}$ and $\alpha_{t_1} = \alpha_{t_0}$. Hence $A_{t_1} = A_s$ and $\alpha_{t_1} = \alpha_s$. Repeating this procedure, we get finally $A_t = A_s$ and $\alpha_t = \alpha_s$. □

Combining Lemma 2.4 with Lemmas 2.2 and 2.3, one can easily prove the following two analogues of Lemma 2.5, respectively.

**Lemma 2.6.** *Assume that $U$ is a nonempty open and connected subset of $X$ and $f : U \to \mathbb{C}$. Let $\{U_t : t \in T\}$ be a family of nonempty open sets such that $U = \bigcup_{t\in T} U_t$. Assume that, for every $t \in T$, there exist a nonconstant exponential function $\phi_t : X \to \mathbb{C}$ and $\alpha_t, \beta_t \in \mathbb{C}$ such that*

$$f(x) = \alpha_t \phi_t(x) + \beta_t \quad \text{for} \quad x \in U_t. \tag{15}$$

*Then, for every $s, t \in T$, we have $\alpha_s = \alpha_t$ and $\beta_s = \beta_t$. Furthermore, if $\alpha_{t_0} \neq 0$ for some $t_0 \in T$ then $\phi_s = \phi_t$ for $s, t \in T$.*

**Lemma 2.7.** *Let $U$ be a nonempty open and connected subset of $X$, $\{U_t : t \in T\}$ be a family of nonempty open sets such that $U = \bigcup_{t\in T} U_t$ and let $f : U \to \mathbb{C}$. Assume that, for every $t \in T$, one of the following two possibilities holds:*

(a) *there exist an additive function $A_t : X \to \mathbb{C}$ and a constant $a_t \in \mathbb{C}$ such that (14) is valid,*

(b) *there exist a nonconstant exponential function* $\phi_t : X \to \mathbb{C} \setminus \{0\}$ *and constants* $\alpha_t \in \mathbb{C} \setminus \{0\}$, $\beta_t \in \mathbb{C}$ *such that* (15) *is valid.*

*Then either* $(a)$ *holds for every* $t \in T$, *or* $(b)$ *holds for every* $t \in T$.

**Proposition 2.8.** *Let* $D$ *be a nonempty and open subset of* $X^2$ *such that the set* $D_+$ *is connected. Assume that a quadruple* $(k, l, m, n)$, *where* $k : D_+ \to \mathbb{C}$, $l, m : D_1 \to \mathbb{C}$ *and* $n : D_2 \to \mathbb{C}$, *satisfies Eq.* (3)*. Then one of the following possibilities holds:*

(a) *there exist an additive function* $A : X \to \mathbb{C}$ *and a constant* $a \in \mathbb{C}$ *such that*

$$k(z) = A(z) + a \quad \text{for} \quad z \in D_+, \tag{16}$$

(b) *there exist a nonconstant exponential function* $\phi : X \to \mathbb{C} \setminus \{0\}$ *and constants* $\alpha \in \mathbb{C} \setminus \{0\}$, $\beta \in \mathbb{C}$ *such that*

$$k(z) = \alpha\phi(z) + \beta \quad \text{for} \quad z \in D_+. \tag{17}$$

*Proof.* Since $D$ is open, for every $(u, v) \in D$ there is an open ball $B_{(u,v)} \subset X$ centered at 0 such that $(u + B_{(u,v)}) \times (v + B_{(u,v)}) \subset D$. Thus, in view of (3), for every $(u, v) \in D$, we get

$$k(x + y) = l(x) + m(x)n(y) \quad \text{for} \quad (x, y) \in (u + B_{(u,v)}) \times (v + B_{(u,v)}). \tag{18}$$

Let $U_{(u,v)} = u + v + B_{(u,v)} + B_{(u,v)}$ for $(u, v) \in D$. Applying [6, Proposition 1], we obtain that, for every $(u, v) \in D$, one of the following possibilities holds:

(i) there exist an additive function $A_{(u,v)} : X \to \mathbb{C}$ and an $a_{(u,v)} \in \mathbb{C}$ such that

$$k(z) = A_{(u,v)}(z) + a_{(u,v)} \quad \text{for} \quad z \in U_{(u,v)},$$

(ii) there exist a nonconstant exponential function $\phi_{(u,v)} : X \to \mathbb{C}$ and $\alpha_{(u,v)} \in \mathbb{C} \setminus \{0\}$, $\beta_{(u,v)} \in \mathbb{C}$ such that

$$k(z) = \alpha_{(u,v)}\phi_{(u,v)}(z) + \beta_{(u,v)} \quad \text{for} \quad z \in U_{(u,v)}.$$

Since $D_+$ is connected and $D_+ = \bigcup_{(u,v)\in D} U_{(u,v)}$, in virtue of Lemma 2.7, either $(i)$ is valid for every $(u, v) \in D$, or $(ii)$ holds for every $(u, v) \in D$. In the first case, applying Lemma 2.5, we obtain $(a)$. In the latter case, making use of Lemma 2.6, we get $(b)$. □

Since every nonzero additive function as well as every nonconstant exponential function, mapping $X$ into $\mathbb{C}$, are locally nonconstant, from Proposition 2.8 we obtain the following result, which generalizes [6, Proposition 2].

**Corollary 2.9.** *Let* $D$ *be a nonempty and open subset of* $X^2$ *such that the set* $D_+$ *is connected. Assume that a quadruple* $(k, l, m, n)$, *where* $k : D_+ \to \mathbb{C}$, $l, m : D_1 \to \mathbb{C}$ *and* $n : D_2 \to \mathbb{C}$, *satisfies Eq.* (3)*. Then either* $k$ *is constant, or it is locally nonconstant.*

## 3. Main Results

We begin this section with the result describing the solutions of (3).

**Theorem 3.1.** *Let $D$ be a nonempty and open subset of $X^2$ such that the set $D_+$ and at least one of the sets $D_1$, $D_2$, are connected. Assume that a quadruple of functions $(k, l, m, n)$, where $k : D_+ \to \mathbb{C}$, $l, m : D_1 \to \mathbb{C}$ and $n : D_2 \to \mathbb{C}$, satisfies Eq. (3). If $k$ is nonconstant then one of the following alternatives holds:*

(a) *there exist a nonzero additive function $A : X \to \mathbb{C}$ and $b, c \in \mathbb{C}$, $d \in \mathbb{C} \setminus \{0\}$ such that*

$$\begin{cases} k(z) = A(z) + b + c & \text{for } z \in D_+, \\ l(x) = A(x) + b & \text{for } x \in D_1, \\ m(x) = d & \text{for } x \in D_1, \\ n(y) = d^{-1}(A(y) + c) & \text{for } y \in D_2; \end{cases} \tag{19}$$

(b) *there exist a nonconstant exponential function $\phi : X \to \mathbb{C}$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\gamma, \delta \in \mathbb{C}$ such that*

$$\begin{cases} k(z) = \alpha\beta\phi(z) + \gamma & \text{for } z \in D_+, \\ l(x) = -\alpha\delta\phi(x) + \gamma & \text{for } x \in D_1, \\ m(x) = \alpha\phi(x) & \text{for } x \in D_1, \\ n(y) = \beta\phi(y) + \delta & \text{for } y \in D_2. \end{cases} \tag{20}$$

*Conversely, if one of the alternatives (a)–(b) holds then the quadruple $(k, l, m, n)$ satisfies (3).*

*Proof.* Assume that $k$ is nonconstant. Since $D_+$ is connected, applying Corollary 2.9, we get that $k$ is locally nonconstant. Furthermore, as $D$ is open, for every $(u, v) \in D$ there is an open ball $B_{(u,v)}$ centered at 0 such that $(u + B_{(u,v)}) \times (v + B_{(u,v)}) \subset D$. Thus, in view of (3), for every $(u, v) \in D$, (18) is valid and so, applying [6, Proposition 1], we obtain that, for every $(u, v) \in D$, either

(i) there exist a unique nonzero additive function $A_{(u,v)} : X \to \mathbb{C}$ and unique constants $b_{(u,v)}, c_{(u,v)} \in \mathbb{C}$ and $d_{(u,v)} \in \mathbb{C} \setminus \{0\}$, such that

$$\begin{cases} k(z) = A_{(u,v)}(z) + b_{(u,v)} + c_{(u,v)} & \text{for } z \in u + v + B_{(u,v)} + B_{(u,v)}, \\ l(x) = A_{(u,v)}(x) + b_{(u,v)} & \text{for } x \in u + B_{(u,v)}, \\ m(x) = d_{(u,v)} & \text{for } x \in u + B_{(u,v)}, \\ n(y) = d_{(u,v)}^{-1}(A_{(u,v)}(y) + c_{(u,v)}) & \text{for } y \in v + B_{(u,v)}; \end{cases}$$

or

(ii) there exist a unique nonconstant exponential function $\phi_{(u,v)}: X \to \mathbb{C}$ and unique constants $\alpha_{(u,v)}, \beta_{(u,v)} \in \mathbb{C} \setminus \{0\}$, $\gamma_{(u,v)}, \delta_{(u,v)} \in \mathbb{C}$ such that

$$\begin{cases} k(z) = \alpha_{(u,v)}\beta_{(u,v)}\phi_{(u,v)}(z) + \gamma_{(u,v)} & \text{for } z \in u+v+B_{(u,v)}+B_{(u,v)}, \\ l(x) = -\alpha_{(u,v)}\delta_{(u,v)}\phi_{(u,v)}(x) + \gamma_{(u,v)} & \text{for } x \in u+B_{(u,v)}, \\ m(x) = \alpha_{(u,v)}\phi_{(u,v)}(x) & \text{for } x \in u+B_{(u,v)}, \\ n(y) = \beta_{(u,v)}\phi_{(u,v)}(y) + \delta_{(u,v)} & \text{for } y \in v+B_{(u,v)}. \end{cases}$$

Since

$$D = \bigcup_{(u,v)\in D} (u+B_{(u,v)}) \times (v+B_{(u,v)}), \tag{21}$$

we have

$$D_1 = \bigcup_{(u,v)\in D} (u+B_{(u,v)}), \tag{22}$$

$$D_2 = \bigcup_{(u,v)\in D} (v+B_{(u,v)}) \tag{23}$$

and

$$D_+ = \bigcup_{(u,v)\in D} (u+v+B_{(u,v)}+B_{(u,v)}). \tag{24}$$

Thus, as $D_+$ is connected and, for every $(u,v) \in D$, $k$ has one of the forms given in (i)–(ii), according to Lemma 2.7, either (i) holds for every $(u,v) \in D$, or (ii) holds for every $(u,v) \in D$.

First, assume that (i) is valid for every $(u,v) \in D$. Fix an $(u_0,v_0) \in D$ and set $A := A_{(u_0,v_0)}$, $b := b_{(u_0,v_0)}$, $c := c_{(u_0,v_0)}$ and $d := d_{(u_0,v_0)}$. Note that $A: X \to \mathbb{C}$ is a nonzero additive function and $d \neq 0$, so in order to prove $(a)$, it is sufficient to show that

$$A_{(u,v)} = A \ \text{ for } \ (u,v) \in D, \tag{25}$$

$$b_{(u,v)} = b \ \text{ for } \ (u,v) \in D, \tag{26}$$

$$c_{(u,v)} = c \ \text{ for } \ (u,v) \in D \tag{27}$$

and

$$d_{(u,v)} = d \ \text{ for } \ (u,v) \in D. \tag{28}$$

In view of $(i)$, for every $(u,v) \in D$ we have

$$k(z) = A_{(u,v)}(z) + b_{(u,v)} + c_{(u,v)} \ \text{ for } \ z \in u+v+B_{(u,v)}+B_{(u,v)},$$

where $A_{(u,v)}$ is an additive function. Thus, as $D_+$ is connected, making use of (24) and applying Lemma 2.5, we obtain (25) and

$$b_{(u,v)} + c_{(u,v)} = b + c \ \text{ for } \ (u,v) \in D. \tag{29}$$

Suppose that $D_2$ is connected. From $(i)$ and (25), for every $(u,v) \in D$, we derive that

$$n(y) = d_{(u,v)}^{-1} A(y) + d_{(u,v)}^{-1} c_{(u,v)} \ \text{ for } \ y \in v+B_{(u,v)}.$$

Obviously, for every $(u, v) \in D$, $d_{(u,v)}^{-1} A$ is an additive function. Therefore, taking into account (23) and applying Lemma 2.5, we get

$$d_{(u,v)}^{-1} A = d^{-1} A \ \text{ for } \ (u, v) \in D \tag{30}$$

and

$$d_{(u,v)}^{-1} c_{(u,v)} = d^{-1} c \ \text{ for } \ (u, v) \in D. \tag{31}$$

Since $A \neq 0$, from (30) we deduce (28). Then, from (28) and (31), we derive (27). Therefore, making use of (29), we get (26). Thus (25)–(28) hold.

Next, assume that $D_1$ is connected. Taking into account (22) and the forms of $l$ and $m$ in (i), in virtue of Lemma 2.5, we obtain (26) and (28), respectively. Hence, making use of (29), we get (27), and so we have (25)–(28).

Now, consider the case where (ii) holds for every $(u, v) \in D$. Fix an $(u_0, v_0) \in D$ and put $\phi := \phi_{(u_0,v_0)}$, $\alpha := \alpha_{(u_0,v_0)}$, $\beta := \beta_{(u_0,v_0)}$, $\gamma := \gamma_{(u_0,v_0)}$ and $\delta := \delta_{(u_0,v_0)}$. Then $\phi : X \to \mathbb{C}$ is a nonconstant exponential function and $\alpha, \beta \neq 0$. So, in order to prove $(b)$, it is enough to show that

$$\phi_{(u,v)} = \phi \ \text{ for } \ (u, v) \in D, \tag{32}$$

$$\alpha_{(u,v)} = \alpha \ \text{ for } \ (u, v) \in D, \tag{33}$$

$$\beta_{(u,v)} = \beta \ \text{ for } \ (u, v) \in D, \tag{34}$$

$$\gamma_{(u,v)} = \gamma \ \text{ for } \ (u, v) \in D \tag{35}$$

and

$$\delta_{(u,v)} = \delta \ \text{ for } \ (u, v) \in D. \tag{36}$$

Taking into account (ii), for every $(u, v) \in D$ we have

$$k(z) = \alpha_{(u,v)} \beta_{(u,v)} \phi_{(u,v)}(z) + \gamma_{(u,v)} \ \text{ for } \ z \in u + v + B_{(u,v)} + B_{(u,v)}.$$

Observe that, for every $(u, v) \in D$, $\phi_{(u,v)}$ is a nonconstant exponential function and $\alpha_{(u,v)} \beta_{(u,v)} \neq 0$. Therefore, as $D_+$ is connected, taking into account (24) and applying Lemma 2.6, we get (32), (35) and

$$\alpha_{(u,v)} \beta_{(u,v)} = \alpha\beta \ \text{ for } \ (u, v) \in D. \tag{37}$$

Suppose that $D_1$ is connected. Then, according to (ii), (32) and (35), for every $(u, v) \in D$, we have

$$m(x) = \alpha_{(u,v)} \phi(x) \ \text{ for } \ x \in u + B_{(u,v)}$$

and

$$l(x) = -\alpha_{(u,v)} \delta_{(u,v)} \phi(x) + \gamma \ \text{ for } \ x \in u + B_{(u,v)}.$$

Thus, making use of Lemma 2.6, in view of (22), we obtain (33) and

$$-\alpha_{(u,v)} \delta_{(u,v)} = -\alpha\delta \ \text{ for } \ (u, v) \in D. \tag{38}$$

Since $\alpha \neq 0$, according to (33), from (37) and (38), we derive (34) and (36), respectively. Consequently, (32)–(36) hold.

Assume that $D_2$ is connected. Then, in view of $(ii)$ and (32), for every $(u, v) \in D$, we have

$$n(y) = \beta_{(u,v)}\phi(y) + \delta_{(u,v)} \quad \text{for} \quad y \in v + B_{(u,v)}. \tag{39}$$

Hence, taking into account (23) and applying Lemma 2.6, we obtain (34) and (36). So, as $\beta \neq 0$, (34) and (37) imply (33). Thus, we get (32)–(36) again.

The converse is easy to check. □

From Theorem 3.1 we derive the following result, which generalizes [5, Corollary].

**Corollary 3.2.** *Let $D \subset \mathbb{R}^2$ be nonempty, open and such that the set $D_+$ and at least one of the sets $D_1$, $D_2$, are connected. A quadruple of functions $(k, l, m, n)$, where $k : D_+ \to \mathbb{R}$ is nonconstant and measurable, $l, m : D_1 \to \mathbb{R}$ and $n : D_2 \to \mathbb{R}$, satisfies Eq.* (3) *if and only if one of the following alternatives holds:*

(a) *there exist $a, d \in \mathbb{R} \setminus \{0\}$ and $b, c \in \mathbb{R}$ such that*

$$\begin{cases} k(z) = az + b + c & \text{for } z \in D_+, \\ l(x) = ax + b & \text{for } x \in D_1, \\ m(x) = d & \text{for } x \in D_1, \\ n(y) = \frac{1}{d}(ay + c) & \text{for } y \in D_2; \end{cases}$$

(b) *there exist $a, \alpha, \beta \in \mathbb{R} \setminus \{0\}$ and $\gamma, \delta \in \mathbb{R}$ such that*

$$\begin{cases} k(z) = \alpha\beta e^{az} + \gamma & \text{for } z \in D_+, \\ l(x) = -\alpha\delta e^{ax} + \gamma & \text{for } x \in D_1, \\ m(x) = \alpha e^{ax} & \text{for } x \in D_1, \\ n(y) = \beta e^{ay} + \delta & \text{for } y \in D_2. \end{cases}$$

*Proof.* Assume that $k : D_+ \to \mathbb{R}$ is nonconstant and measurable, $l, m : D_1 \to \mathbb{R}$, $n : D_2 \to \mathbb{R}$ and a quadruple $(k, l, m, n)$ satisfies (3). Then, applying Theorem 3.1, we obtain that one of the following possibilities holds:

(i) there exist a nonzero additive function $A : \mathbb{R} \to \mathbb{C}$ and $b, c \in \mathbb{C}$, $d \in \mathbb{C} \setminus \{0\}$ such that the quadruple $(k, l, m, n)$ is of the form (19);

(ii) there exist a nonconstant exponential function $\phi : \mathbb{R} \to \mathbb{C}$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\gamma, \delta \in \mathbb{C}$ such that the quadruple $(k, l, m, n)$ is of the form (20).

In the case of $(i)$, we get $d \in \mathbb{R} \setminus \{0\}$ and

$$A(x - y) = A(x) - A(y) = l(x) - l(y) \in \mathbb{R} \quad \text{for} \quad x, y \in D_1.$$

Hence $A(D_1 - D_1) \subset \mathbb{R}$ and so, as $D_1 - D_1$ is a nonempty open subset of $\mathbb{R}$, we get $A : \mathbb{R} \to \mathbb{R}$. Thus, taking into account (19), we conclude that also $b, c \in \mathbb{R}$. Finally, since $k$ is measurable, so is $A$. Therefore (cf. [10, Theorems 5.5.2 and 9.4.3]) $A(x) = ax$ for $x \in \mathbb{R}$, with some $a \in \mathbb{R} \setminus \{0\}$. Hence, $(a)$ is valid.

In the case of $(ii)$, we have $m(x) = \alpha\phi(x) \neq 0$ for $x \in D_1$, because $\alpha \neq 0$ and $\phi$ is a nonconstant exponential function. Thus, in view of (20), we get $\phi(x - y) = \frac{m(x)}{m(y)} \in \mathbb{R} \setminus \{0\}$ for $x, y \in D_1$. Since $D_1 - D_1$ is a nonempty open subset of $\mathbb{R}$, this implies that $\phi : \mathbb{R} \to \mathbb{R}$. Moreover, taking an $x \in D_1$, by (20), we get $\alpha = \frac{m(x)}{\phi(x)} \in \mathbb{R} \setminus \{0\}$. Note also that $\phi$, being nonconstant, is locally nonconstant. Thus there exist $x, y \in D_2$ with $\phi(x) \neq \phi(y)$. Therefore, making use of (20) again, we obtain that $\beta = \frac{n(x)-n(y)}{\phi(x)-\phi(y)} \in \mathbb{R} \setminus \{0\}$ and so $\gamma, \delta \in \mathbb{R}$. Furthermore, since $k$ is measurable, so is $\phi$. Hence (cf. [10, Theorems 13.1.4 and 13.1.7]) $\phi(x) = e^{ax}$ for $x \in \mathbb{R}$, with some $a \in \mathbb{R} \setminus \{0\}$. In this way we have proved that $(b)$ holds.

The converse is easy to check. □

The next result concerns the existence and uniqueness of extension of solutions of Eq. (3).

**Theorem 3.3.** *Let $D$ be a nonempty and open subset of $X^2$ such that the set $D_+$ and at least one of the sets $D_1$, $D_2$, are connected. Assume that a quadruple of functions $(k, l, m, n)$, where $k : D_+ \to \mathbb{C}$ is nonconstant, $l, m : D_1 \to \mathbb{C}$ and $n : D_2 \to \mathbb{C}$, satisfies Eq. (3). Then there exists a unique quadruple $(K, L, M, N)$ of functions mapping $X$ into $\mathbb{C}$ such that (6) and (7) hold.*

*Proof.* According to Theorem 3.1, either there exist a nonzero additive function $A : X \to \mathbb{C}$ and $b, c \in \mathbb{C}$, $d \in \mathbb{C} \setminus \{0\}$ such that (19) is valid, or there exist a nonconstant exponential function $\phi : X \to \mathbb{C}$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\gamma, \delta \in \mathbb{C}$ such that (20) holds. In the first case, a quadruple of functions $(K, L, M, N)$, mapping $X$ into $\mathbb{C}$, given by $K(x) = A(x) + b + c$ for $x \in X$, $L(x) = A(x) + b$ for $x \in X$, $M(x) = d$ for $x \in X$ and $N(x) = d^{-1}(A(x) + c)$ for $x \in X$, satisfies (6) and (7). In order to prove the uniqueness, suppose that a quadruple $(\tilde{K}, \tilde{L}, \tilde{M}, \tilde{N})$ of functions mapping $X$ into $\mathbb{C}$ satisfies (6) and (7). Then $\tilde{M}(x) = m(x) = d$ for $x \in D_1$. Therefore, applying Theorem 3.1 with $D = X^2$, we conclude that $\tilde{M} = d$ and there exist a nonzero additive function $\tilde{A} : X \to \mathbb{C}$ and $\tilde{b}, \tilde{c} \in \mathbb{C}$ such that $\tilde{L}(x) = \tilde{A}(x) + \tilde{b}$ for $x \in X$ and $\tilde{N}(x) = d^{-1}(\tilde{A}(x) + \tilde{c})$ for $x \in X$. So, taking into account Lemma 2.1, we obtain that $\tilde{A} = A$, $\tilde{b} = b$ and $\tilde{c} = c$, which proves the uniqueness of the extension.

In the second case similar arguments work. □

The following example shows that the connectedness of the sets $D_1$ and $D_2$, in general, is not sufficient for the existence of the extension of a solution of (3) to a solution of (5).

*Example* 3.4. Let $D = (0, \infty)^2 \cup \{(x, y) \in \mathbb{R} : x + y < 0\}$. Then $D_+ = (-\infty, 0) \cup (0, \infty)$ and $D_1 = D_2 = \mathbb{R}$. Define $k : (-\infty, 0) \cup (0, \infty) \to \mathbb{C}$ and $l, m, n : \mathbb{R} \to \mathbb{C}$ in the following way:

$$k(x) = \begin{cases} 1 & \text{for } \; x > 0, \\ 0 & \text{for } \; x < 0, \end{cases}$$

$$l(x) = 0 \quad \text{for} \quad x \in \mathbb{R}$$

and

$$m(x) = n(x) = \begin{cases} 1 & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x \le 0. \end{cases}$$

Then $k$ is nonconstant and, as a straightforward calculation shows, the quadruple $(k, l, m, n)$ satisfies (3). On the other hand, if $(K, L, M, N)$ were the extension of $(k, l, m, n)$ to the solution of (5) on $\mathbb{R}^2$, then we would have

$$1 = K(1) = L(2) + M(2)N(-1) = 0,$$

which gives a contradiction.

We conclude the paper with two results concerning the solutions of a particular case of Eq. (3), namely

$$k(x + y) = l(x) + m(x)k(y) \quad \text{for} \quad (x, y) \in D. \tag{40}$$

**Theorem 3.5.** *Let $D$ be a nonempty and open subset of $X^2$ such that $D_+$ and $D_+ \cup D_2$ are connected. Assume that a triple of functions $(k, l, m)$, where $k : D_+ \cup D_2 \to \mathbb{C}$ and $l, m : D_1 \to \mathbb{C}$, satisfies Eq. (40). If $k_{|D_+}$ is nonconstant then one of the following alternatives holds:*

(a) *there exist a nonzero additive function $A : X \to \mathbb{C}$ and a constant $a \in \mathbb{C}$ such that*
$$\begin{cases} k(z) = A(z) + a & \text{for } z \in D_+ \cup D_2, \\ l(x) = A(x) & \text{for } x \in D_1, \\ m(x) = 1 & \text{for } x \in D_1; \end{cases}$$

(b) *there exist a nonconstant exponential function $\phi : X \to \mathbb{C}$, an $\alpha \in \mathbb{C} \setminus \{0\}$ and a $\beta \in \mathbb{C}$ such that*
$$\begin{cases} k(z) = \alpha\phi(z) + \beta & \text{for } z \in D_+ \cup D_2, \\ l(x) = \beta(1 - \phi(x)) & \text{for } x \in D_1, \\ m(x) = \phi(x) & \text{for } x \in D_1. \end{cases}$$

*Conversely, if one of the alternatives (a)–(b) holds then the triple $(k, l, m)$ satisfies (40).*

*Proof.* Assume that $k$ is nonconstant on $D_+$. Then, according to Proposition 2.8, either there exist a nonzero additive function $A : X \to \mathbb{C}$ and a constant $a \in \mathbb{C}$ such that (16) holds, or there exist a nonconstant exponential function $\phi : X \to \mathbb{C} \setminus \{0\}$ and constants $\alpha \in \mathbb{C} \setminus \{0\}$, $\beta \in \mathbb{C}$ such that (17) is valid. Moreover, as $D$ is open, for every $y \in D_2 \setminus D_+$, there exist an $x \in X$ and an open ball $B_y \subset X$ centered at 0 such that $D_{(x,y)} := (x, y) + B_y \times B_y \subset D$. Furthermore, as $D_{(x,y)}$ is a nonempty open and connected subset of $X^2$, the sets $(D_{(x,y)})_+$ and $(D_{(x,y)})_2 = y + B_y$ are connected. Hence, taking into account (40) and applying Theorem 3.1, we obtain that either there exist a nonzero additive function $A_y : X \to \mathbb{C}$, a $c_y \in \mathbb{C}$ and a $d_y \in \mathbb{C} \setminus \{0\}$ such that $k(v) = d_y^{-1}(A_y(v) + c_y)$ for $v \in y + B_y$, or there exist a nonconstant exponential function $\phi_y : X \to \mathbb{C}$, a $\beta_y \in \mathbb{C} \setminus \{0\}$ and a $\delta_y \in \mathbb{C}$ such that $k(v) = \beta_y\phi_y(v) + \delta_y$ for $v \in y + B_y$. Therefore, as $D_+ \cup D_2$ is a nonempty, open and connected subset of $X^2$ and $D_+ \cup D_2 = D_+ \cup \bigcup_{y \in D_2 \setminus D_+} (y + B_y)$, applying Lemma 2.7 and then Lemmas 2.5 and 2.6, we obtain that either $k(z) = A(z) + a$ for $z \in D_+ \cup D_2$, or $k(z) = \alpha\phi(z) + \beta$ for $z \in D_+ \cup D_2$, respectively. If the first possibility holds, then from (40) we derive that

$$(1 - m(x))(A(y) + a) = l(x) - A(x) \quad \text{for} \quad (x, y) \in D. \tag{41}$$

If $m(x) \neq 1$ for some $x \in D_1$ then taking a $y \in D_2$ with $(x, y) \in D$ and an open ball $B \subset X$ centered at 0 such that $(x, y) + B \times B \subset D$, we get that $A$ is constant on $y + B$. Since $A$ is nonzero additive function, this yields a contradiction. Hence $m(x) = 1$ for $x \in D_1$, which together with (41), gives $l(x) = A(x)$ for $x \in D_1$. Thus, we get $(a)$. If the second possibility is valid, then the similar arguments lead to $(b)$.

The converse is easy to check. □

Applying Theorem 3.5 and repeating the arguments from the proof of Theorem 3.3, we obtain the following result.

**Theorem 3.6.** *Let $D$ be a nonempty and open subset of $X^2$ such that $D_+$ and $D_+ \cup D_2$ are connected. Assume that a triple of functions $(k, l, m)$, where $k : D_+ \cup D_2 \to \mathbb{C}$ and $l, m : D_1 \to \mathbb{C}$, satisfies Eq. (40). If $k_{|D_+}$ is nonconstant then there exists a unique triple $(K, L, M)$ of functions mapping $X$ into $\mathbb{C}$ such that*

$$K(x + y) = L(x) + M(x)K(y) \quad \textit{for} \quad (x, y) \in X^2$$

*and*

$$k = K_{|D_+ \cup D_2}, \quad l = L_{|D_1}, \quad m = M_{|D_1}.$$



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Małgorzata Chudziak and Barbara Sobek
Faculty of Mathematics and Natural Sciences
University of Rzeszów
Pigonia 1
35-310 Rzeszów
Poland
e-mail: mchudziak@ur.edu.pl;
bsobek@ur.edu.pl

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